Homogenization of the wave equation over large distances : Transport, Radiative Transfer and Diffusion

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Outline

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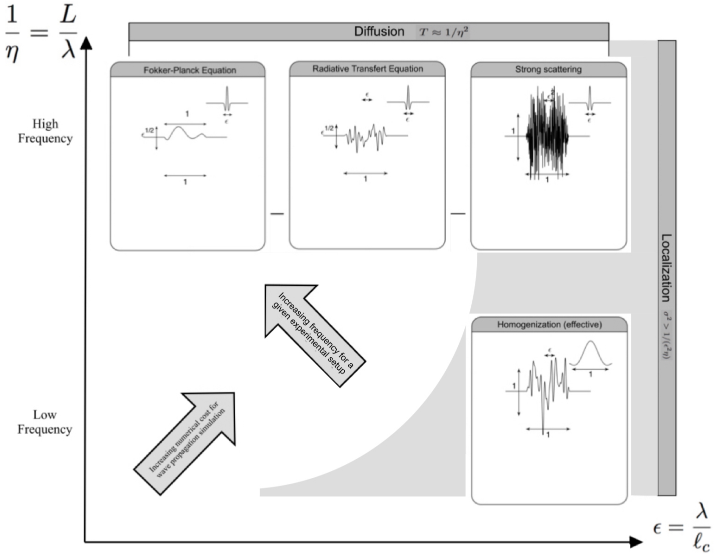
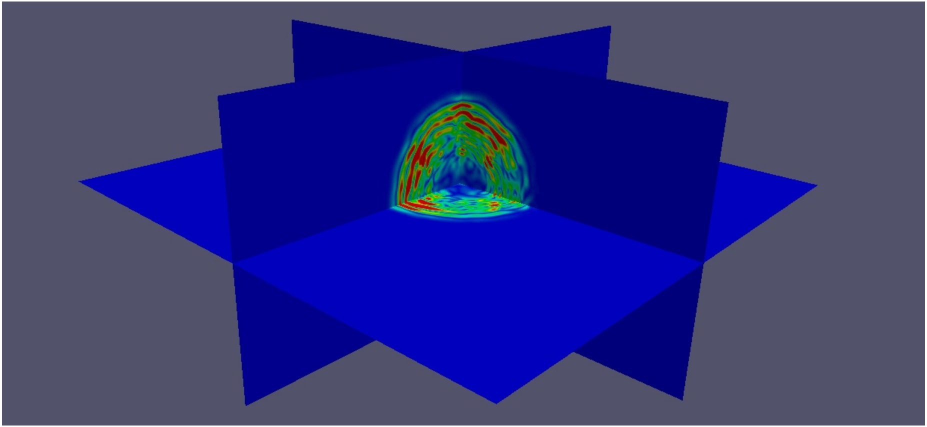
The scaling regimes

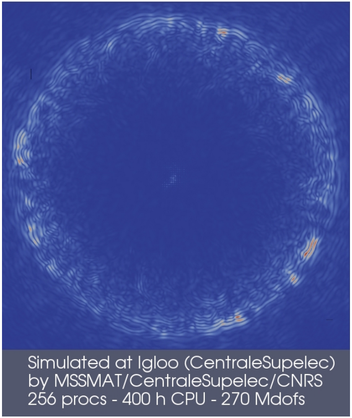


Wave propagation over large distances in a heterogeneous medium

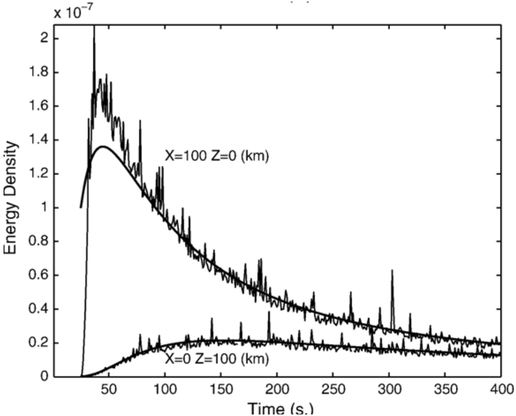
$65 \times 65 \times 33\ \ell_c^3$, 5×10^5 elements, 20 CPU days on 156 processors (on SGI machine with 800 Intel Xeon X5650 cores)



High-frequency regime ($\ell_c \approx \lambda \ll L$, $\sigma \ll 1$)



(a) Snapshot of displacement field



(b) Energy density computed from wave equation and diffusion [1]

- Displacement field $\mathbf{u}(\mathbf{x}, t)$ is not stable between two realizations
- Consideration of energy densities (in phase space $\mathbf{x} \times \mathbf{k}$) : Wigner transform of $\mathbf{u}(\mathbf{x}, t)$

1. L. Margerin. "Attenuation, transport and diffusion of scalar waves in textured random media". In : *Tectonophys.* 416.1-4 (2006), p. 229-244. DOI : 10.1016/j.tecto.2005.11.011

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A first attempt at measuring an "energy" [2]

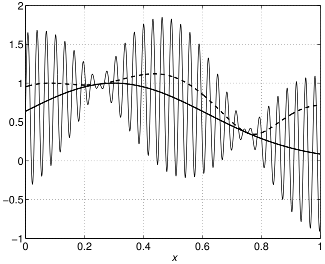

FIGURE – Oscillating function $u_\epsilon(x)$ (thin line), mean function $\underline{u}(x)$ (thick line), and square-root limit $(\underline{u}(x)^2 + \frac{1}{2}a(x)^2)^{\frac{1}{2}}$ (thick dashed line)

Consider a real function $x \to u_\epsilon(x)$ oscillating with amplitude $a(x)$ about its mean $\underline{u}(x)$:

$$u_\epsilon(x) = \underline{u}(x) + a(x)\sin\frac{x}{\epsilon}, \quad 0 < \epsilon \ll 1.$$

has no strong limit when $\epsilon \to 0$, although the functions a and $\underline{u}$ vary slowly. However for any smooth function ϕ with compact support on $\mathbb{R}^3$:

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\phi(x)\,(u_\epsilon(x))^2\,dx = \int_{\mathbb{R}^3}\phi(x)\left((\underline{u}(x))^2 + \frac{1}{2}(a(x))^2\right)dx\,.$$

2. I. BAYDOUN et al. "Kinetic modeling of multiple scattering of elastic waves in heterogeneous anisotropic media". In : *Wave Motion* 51.8 (2014), p. 1325-1348. DOI : 10.1016/j.wavemoti.2014.08.001

The Wigner measure in 1D

- The objective is to define a quadratic quantity rescaled to observe around a certain (high) frequency

$$W[u](x,k) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iky} u\left(x - \frac{\epsilon y}{2}\right) \overline{u\left(x + \frac{\epsilon y}{2}\right)} dy$$

- Examples
  - Constant function : $u(x) = u_0$

$$W[u] = \delta(0)$$

  - Fluctuating function around frequency $\mathcal{O}(1)$: $u(x) = \exp(iqx)$

$$W[u] = \delta(\epsilon q) \to_{\epsilon\to 0} \delta(0)$$

  - Fluctuating function around frequency $\mathcal{O}(1/\epsilon)$: $u(x) = \exp(iqx/\epsilon)$

$$W[u] = \delta(q)$$

The Wigner measure as high-frequency energy density

The Wigner measure

- Wigner transform

$$\boldsymbol{W}_\epsilon[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{x},\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\boldsymbol{u}\left(\boldsymbol{x}-\frac{\epsilon\boldsymbol{y}}{2}\right)\otimes\overline{\boldsymbol{v}\left(\boldsymbol{x}+\frac{\epsilon\boldsymbol{y}}{2}\right)}\,d\boldsymbol{y},$$

- Wigner measure $\boldsymbol{W}[\boldsymbol{u}_\epsilon] = \boldsymbol{W}_\epsilon[\boldsymbol{u}_\epsilon,\boldsymbol{u}_\epsilon]$ is the limit (high-frequency) energy of $(\boldsymbol{u}_\epsilon)$. It is positive in the limit.

- Equivalent definition

$$\boldsymbol{W}_\epsilon[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{x},\boldsymbol{k}) = \int_{\mathbb{R}^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\boldsymbol{u}}\left(\frac{\boldsymbol{p}}{2}+\frac{\boldsymbol{k}}{\epsilon}\right)\otimes\overline{\hat{\boldsymbol{v}}\left(\frac{\boldsymbol{p}}{2}-\frac{\boldsymbol{k}}{\epsilon}\right)}\,d\boldsymbol{p},$$

Among other quadratic quantities, the high-frequency strain energy $\mathcal{E}_\epsilon(t) := \frac{1}{2}\int_D \mathbf{C}\nabla\boldsymbol{u}_\epsilon : \nabla\boldsymbol{u}_\epsilon\,d\boldsymbol{x}$ and kinetic energy $\mathcal{T}_\epsilon(t) := \frac{1}{2}\int_D \rho|\partial_t\boldsymbol{u}_\epsilon|^2\,d\boldsymbol{x}$ can be retrieved from

$$\lim_{\epsilon\to 0}\mathcal{E}_\epsilon(t) = \frac{1}{2}\int_{D\times\mathbb{R}^3}\rho(\boldsymbol{x})\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{k}) : \boldsymbol{W}[\boldsymbol{u}_\epsilon(\cdot,t)](d\boldsymbol{x},d\boldsymbol{k}),$$

$$\lim_{\epsilon\to 0}\mathcal{T}_\epsilon(t) = \frac{1}{2}\int_{D\times\mathbb{R}^3}\rho(\boldsymbol{x})\mathrm{Tr}\,\boldsymbol{W}[\epsilon\partial_t\boldsymbol{u}_\epsilon(\cdot,t)](d\boldsymbol{x},d\boldsymbol{k}).$$

Outline

Acoustic wave equation in a homogeneous medium

The acoustic wave equation in a homogeneous medium can be written

$$\begin{aligned} \overline{\rho}\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p &= 0 \\ \frac{1}{\overline{K}}\frac{\partial p}{\partial t} + \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

The non-vanishing solutions of the dispersion matrix verify

$$\omega_{\pm} = \pm \overline{c}|\boldsymbol{k}|$$

with modes $f_{\pm}(t, \boldsymbol{z}, \boldsymbol{k}) = \sqrt{\overline{\rho}/2}(\boldsymbol{u}(t, \boldsymbol{z}) \cdot \hat{\boldsymbol{k}}) \pm \sqrt{1/2\overline{K}}p(t, \boldsymbol{z})$. The (unscaled) Wigner transform of these modes verify a transport equation

$$\frac{\partial a_+}{\partial t} + \overline{c}\hat{\boldsymbol{k}} \cdot \nabla a_+ = 0$$

Acoustic wave equation in a slowly-fluctuating medium

The acoustic wave equation in a slowly-fluctuating medium can be written

$$\begin{aligned}\rho(\boldsymbol{x})\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p &= 0\\ \frac{1}{K(\boldsymbol{x})}\frac{\partial p}{\partial t} + \nabla\cdot\boldsymbol{u} &= 0\end{aligned}$$

where we assume that the frequency is ω/ϵ and $c^2(\boldsymbol{x}) = K(\boldsymbol{x})/\rho(\boldsymbol{x})$.
The non-vanishing solutions of the dispersion matrix verify

$$\omega_\pm = \pm c(\boldsymbol{x})|\boldsymbol{k}|$$

with modes $f_\pm(t,\boldsymbol{x},\boldsymbol{z},\boldsymbol{k}) = \sqrt{\rho(\boldsymbol{x})/2}(\boldsymbol{u}(t,\boldsymbol{z})\cdot\hat{\boldsymbol{k}}) \pm \sqrt{1/2K(\boldsymbol{x})}p(t,\boldsymbol{z})$. In the high-frequency limit ($\epsilon \to 0$) the Wigner transform of the mode verifies a transport (Liouville) equation

$$\frac{\partial a_+}{\partial t} + c(\boldsymbol{x})\hat{\boldsymbol{k}}\cdot\nabla a_+ - |\boldsymbol{k}|\nabla c(\boldsymbol{x})\cdot\nabla_{\boldsymbol{k}} a_+ = 0$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and

$$a_+(t,\boldsymbol{x},\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} f\left(t,\boldsymbol{x},\boldsymbol{x}-\frac{\boldsymbol{y}}{2},\boldsymbol{k}\right)\overline{f}\left(t,\boldsymbol{x},\boldsymbol{x}+\frac{\boldsymbol{y}}{2},\boldsymbol{k}\right)\,d\boldsymbol{y},$$

We follow here the lines of [3].

3. L. Ryzhik, G. Papanicolaou et J. B. Keller. "Transport equations for elastic and other waves in random media". In : *Wave Motion* 24 (1996), p. 327-370. DOI : 10.1016/S0165-2125(96)00021-2

Sketch of the proof

1. Rewrite the system in a condensed form :

$$A(\boldsymbol{x})\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} + D_j\frac{\partial}{\partial x_j}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} = \mathcal{L}\boldsymbol{q} = \boldsymbol{0}$$

where $A(\boldsymbol{x}) = \mathrm{diag}[\rho(\boldsymbol{x})\ \rho(\boldsymbol{x})\ \rho(\boldsymbol{x})\ 1/K(\boldsymbol{x})]$, $D_j = 2\boldsymbol{e}_j \otimes_S \boldsymbol{e}_4$, and the dispersion matrix $\Gamma(\boldsymbol{x}, \boldsymbol{k}) = A^{-1}(\boldsymbol{x})k_j D_j$ is

$$L(\boldsymbol{x}, \boldsymbol{k}) = \begin{bmatrix} 0 & 0 & 0 & k_1/\rho(\boldsymbol{x}) \\ 0 & 0 & 0 & k_2/\rho(\boldsymbol{x}) \\ 0 & 0 & 0 & k_3/\rho(\boldsymbol{x}) \\ k_1 K(\boldsymbol{x}) & k_2 K(\boldsymbol{x}) & k_3 K(\boldsymbol{x}) & 0 \end{bmatrix}$$

whose non-vanishing eigenvalues and eigenvectors are $\omega_\pm = \pm c(\boldsymbol{x})|\boldsymbol{k}|$, and $\boldsymbol{b}_\pm(\boldsymbol{x}, \boldsymbol{k}) = [\frac{\hat{\boldsymbol{k}}}{\sqrt{2\rho(\boldsymbol{x})}},\ \pm\sqrt{\frac{K(\boldsymbol{x})}{2}}]$.

Sketch of the proof – 2

1. Estimate $\boldsymbol{W}_\epsilon[\mathcal{L}\boldsymbol{q}_\epsilon, \boldsymbol{q}_\epsilon] = 0$ and $\boldsymbol{W}_\epsilon[\boldsymbol{q}_\epsilon, \mathcal{L}\boldsymbol{q}_\epsilon] = 0$
2. Expanding the above equations in ϵ yields

$$\frac{\partial W_\epsilon}{\partial t} + (\mathcal{Q}_1^0 + \epsilon \mathcal{Q}_1^1 + \ldots) W_\epsilon + \frac{1}{\epsilon}(\mathcal{Q}_2^0 + \epsilon \mathcal{Q}_2^1 + \ldots) W_\epsilon = 0$$

   where

$$W_\epsilon(t, \boldsymbol{x}, \boldsymbol{k}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \boldsymbol{u}_\epsilon \left(t, \boldsymbol{x} - \frac{\epsilon \boldsymbol{y}}{2}, \boldsymbol{k}\right) \overline{\boldsymbol{u}_\epsilon} \left(t, \boldsymbol{x} + \frac{\epsilon \boldsymbol{y}}{2}, \boldsymbol{k}\right) d\boldsymbol{y},$$

3. Expand the Wigner matrix in series $W_\epsilon = W^{(0)} + \epsilon W^{(1)} + \ldots$
4. The limit Wigner matrix must verify $\mathcal{Q}_2^0 W^{(0)} = 0$, which means it should be projected on the modes of $L(\boldsymbol{x}, \boldsymbol{k})$.
5. The next term should verify

$$\mathcal{Q}_2^0 W^{(1)} = -\frac{\partial W^{(0)}}{\partial t} - (\mathcal{Q}_1^0 + \mathcal{Q}_2^1) W^{(0)}$$

6. Projection on the modes and a solvability argument on the right hand side of the above equation yield the result.

Outline

1. Observation of the acoustic wave field at large distances
   - The acoustic wave equation in the high-frequency regime
   - The Wigner transform
2. Transport equation in a slowly-fluctuating medium
3. Radiative Transfer equation in rapidly-fluctuating medium
4. Diffusion equation at long times
5. Extension : elasticity and anisotropy

Acoustic wave equation in a slowly-fluctuating medium

The acoustic wave equation in a slowly-fluctuating medium can be written

$$\mathcal{A}(\boldsymbol{x})\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} + D_j\frac{\partial}{\partial x_j}\begin{bmatrix}\boldsymbol{u}\\ p\end{bmatrix} = \mathcal{L}\boldsymbol{q} = \boldsymbol{0}$$

where we assume that the frequency is ω/ϵ.
We consider properties that are such that

$$\mathcal{A}(\boldsymbol{x}) = \begin{bmatrix}\overline{\rho}\mathbf{I}_3 & 0\\ 0 & 1/\overline{K}\end{bmatrix}\left(\begin{bmatrix}\mathbf{I}_3 & 0\\ 0 & 1\end{bmatrix} + \sqrt{\epsilon}\begin{bmatrix}\nu_\rho\left(\frac{\boldsymbol{x}}{\epsilon}\right)\mathbf{I}_3 & 0\\ 0 & \nu_K\left(\frac{\boldsymbol{x}}{\epsilon}\right)\end{bmatrix}\right)$$

where $\nu_\rho(\boldsymbol{x})$ and $\nu_K(\boldsymbol{x})$ are zero-mean stationary random fields with mean-zero and covariance functions

$$\mathbf{R}_{\rho\rho}(\boldsymbol{z}) = \mathbb{E}\left[\nu_\rho(\boldsymbol{y})\nu_\rho(\boldsymbol{y}+\boldsymbol{z})\right], \quad \mathbf{R}_{\rho K}(\boldsymbol{z}) = \mathbb{E}\left[\nu_\rho(\boldsymbol{y})\nu_K(\boldsymbol{y}+\boldsymbol{z})\right], \quad \mathbf{R}_{KK}(\boldsymbol{z}) = \mathbb{E}\left[\nu_K(\boldsymbol{y})\nu_K(\boldsymbol{y}+\boldsymbol{z})\right].$$

Radiative Transfer Equation

In the weak scattering limit ($\epsilon \to 0$), the Wigner transform of the mode of the background verifies a radiative transfer equation

$$\frac{\partial a}{\partial t} + c(\boldsymbol{x})\hat{\boldsymbol{k}} \cdot \nabla a - |\boldsymbol{k}|\nabla c(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{k}} a = \int_{\mathbb{R}^3} (a(\boldsymbol{k}') - a(\boldsymbol{k}))\sigma(\boldsymbol{k}, \boldsymbol{k}')d\boldsymbol{k}'$$

where the differential scattering cross-section is

$$\sigma(\boldsymbol{k}, \boldsymbol{k}') = \frac{\pi c(\boldsymbol{x})^2 |\boldsymbol{k}|^2}{2} \Big((\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}')^2 \hat{\mathsf{R}}_{\rho\rho}(\boldsymbol{k} - \boldsymbol{k}') + 2(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}')\hat{\mathsf{R}}_{\rho K}(\boldsymbol{k} - \boldsymbol{k}') + \hat{\mathsf{R}}_{KK}(\boldsymbol{k} - \boldsymbol{k}') \Big) \delta(c(\boldsymbol{x})|\boldsymbol{k}| - c(\boldsymbol{x})|\boldsymbol{k}'|)$$

Radiative Transfer Equation

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- Sketch of proof
  - Multiscale expansion as before ...
  - Ensemble averages are considered at each order.
  - The ϵ^{-1} indicates to project the Wigner measure on the modes of the background dispersion
  - A "mixing" condition is introduced.

Outline

The diffusion equation at long times

We consider for simplicity a homogeneous background $c(\boldsymbol{x}) = \overline{c}$. We consider a different scaling of space and time in the radiative transfer equation : $\boldsymbol{x} \to \boldsymbol{x}/\epsilon$, $t \to t/\epsilon^2$. At long times, the radiative transfer equation reduces to the diffusion equation for an isotropic energy density

$$\frac{\partial}{\partial t} a(t, \boldsymbol{x}, |\boldsymbol{k}|) = \nabla \cdot (D(|\boldsymbol{k}|) \nabla a(t, \boldsymbol{x}, |\boldsymbol{k}|))$$

where the diffusion coefficient is

$$D(|\boldsymbol{k}|) = \frac{\overline{c}^2}{3(\Sigma(|\boldsymbol{k}|) - \lambda(|\boldsymbol{k}|))}$$

and

$$\Sigma(|\boldsymbol{k}|) = \int_{\mathbb{R}^3} \sigma(\boldsymbol{k}, \boldsymbol{k}') d\boldsymbol{k}', \quad \lambda(|\boldsymbol{k}|) = 2\pi \int_{|\boldsymbol{k}| = |\boldsymbol{k}'|} (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}') \sigma(\boldsymbol{k}, \boldsymbol{k}') d\boldsymbol{k}'$$

The diffusion equation at long times – Sketch of proof

- Multiscale expansion in the radiative transfer equation with the new scaling ...

$$\epsilon^2 \frac{\partial}{\partial t}(a_0 + \dots) + \epsilon \overline{c} \hat{\boldsymbol{k}} \cdot \nabla (a_0 + \epsilon a_1 + \dots)$$
$$= \int_{\mathbb{R}^3} ((a_0(\boldsymbol{k}') + \epsilon a_1(\boldsymbol{k}') + \epsilon^2 a_2(\boldsymbol{k}') + \dots) - (a_0(\boldsymbol{k}) + \epsilon a_1(\boldsymbol{k}) + \epsilon^2 a_2(\boldsymbol{k}) + \dots))\sigma(\boldsymbol{k}, \boldsymbol{k}') d\boldsymbol{k}'$$

- Spectral theory indicates that the first order should become isotropic.

$$a_0(t, \boldsymbol{x}, \boldsymbol{k}) = a_0(t, \boldsymbol{x}, |\boldsymbol{k}|)$$

- Next order yields

$$a_1(t, \boldsymbol{x}, \boldsymbol{k}) = -\frac{\overline{c}}{\Sigma(|\boldsymbol{k}|) - \lambda(|\boldsymbol{k}|)} \hat{\boldsymbol{k}} \cdot \nabla a_0(t, \boldsymbol{x}, |\boldsymbol{k}|)$$

- Integration over $\hat{\boldsymbol{k}}$ at the last order yields the diffusion equation for a_0

Outline

The radiative transfer for isotropic elastic waves

Radiative Transfer Equation

Projection of energy density on background modes

$$\partial_t \mathbf{W}(\boldsymbol{x},\boldsymbol{k},t) + \{\omega(\boldsymbol{x},\boldsymbol{k}), \mathbf{W}(\boldsymbol{x},\boldsymbol{k},t)\} = \int \sigma(\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}')\mathbf{W}(\boldsymbol{x},\boldsymbol{k}',t)d\boldsymbol{k}' - \Sigma(\boldsymbol{x},\boldsymbol{k})\mathbf{W}(\boldsymbol{x},\boldsymbol{k},t)$$

for modal energy density matrix $\mathbf{W}$ in $\boldsymbol{x} \times \boldsymbol{k}$ of ω-modes of $\boldsymbol{\Gamma}\boldsymbol{U} = \rho^{-1}\left(\mathrm{C} : \boldsymbol{U} \otimes \boldsymbol{k}\right)\boldsymbol{k}$.

For isotropic homogeneous backgrounds

Projection of energy density on P and S modes

$$\partial_t w_P + c_P \nabla_{\boldsymbol{x}} w_P = \int \sigma_{PP} w_P d\boldsymbol{k}' + \int \sigma_{SP} \mathbf{W}_S d\boldsymbol{k}' - (\Sigma_{PP} + \Sigma_{PS}) w_P$$

$$\partial_t \mathbf{W}_S + c_S \nabla_{\boldsymbol{x}} \mathbf{W}_S = \int \sigma_{PS} w_P d\boldsymbol{k}' + \int \sigma_{SS} \mathbf{W}_S d\boldsymbol{k}' - (\Sigma_{SP} + \Sigma_{SS}) \mathbf{W}_S$$

- Equipartition is predicted by diffusion in elastic media
- The fully anisotropic case can be treated [4]

4. I. Baydoun et al. "Kinetic modeling of multiple scattering of elastic waves in heterogeneous anisotropic media". In : *Wave Motion* 51.8 (2014), p. 1325-1348. DOI : 10.1016/j.wavemoti.2014.08.001

Closing in on the observations : what has dynamic homogenization brought ...

The coherent pulses

- Seem deterministic with an amplitude strongly dependent on distance to source L
- Have strong directionality/anisotropy features
- Are not sensitive to the particular realization of heterogeneity
- Are stronger (relatively to coda) when weak heterogeneities fluctuate faster than wavelength $\lambda \gg \ell_c$ and $\sigma \ll 1$

The coda

- Seems random with an amplitude independent (at late times) on L
- Seems to propagate isotropically
- Is sensitive to the particular realization of heterogeneity
- Is stronger when $\lambda \approx \ell_c$ and $\sigma \approx 1$

- Homogenized models should be able to reproduce these features, random and deterministic

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